Lesson **4-9**

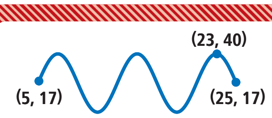
The Graph-Standardization Theorem

▶ **BIG IDEA** The image of the graph of a parent trigonometric function under the composite of a scale change followed by a translation has a predictable equation, amplitude, phase shift, and period.

Recall that a sine wave is the image of the graph of $y = \sin x$ under a composite of translations and scale changes.

In Lessons 4-7 and 4-8, you saw how scale changes affect the amplitude and frequency of sine waves and how translations introduce phase shifts and vertical shifts. In this lesson, you will see how composites of scale changes and translations affect sine waves.

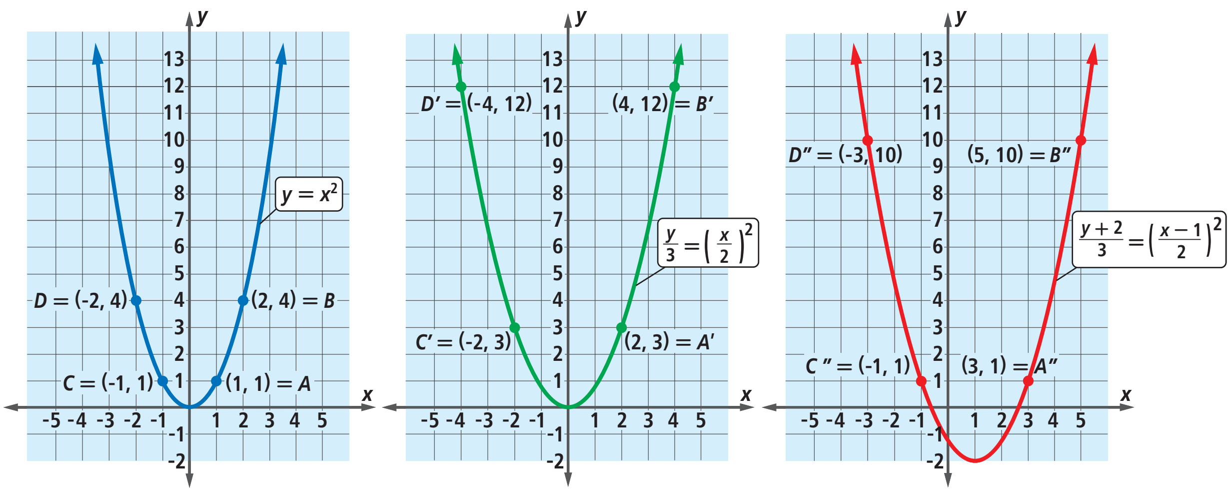
Mental Math



The graph above shows $2\frac{1}{2}$ cycles of a sine wave. What are the period and amplitude of this wave?

A Specific Example

We first apply a composite to a graph that is not a sine wave. Consider the parabola with equation $y = x^2$. Apply the scale change S: $(x, y) = (2x, 3y)$. This stretches the parent graph by a factor of 2 in the x-direction and by a factor of 3 in the y-direction. To this image apply the translation T: $(x, y) \to (x + 1, y - 2)$. This translates the image 1 unit to the right and 2 units down. The graphs below show the order of transformations.



Parent
(–1, 1) (1,1)
(–2, 4) (2, 4)

S(Parent)
(–1, 1) → (–2, 3) (1, 1) → (2, 3)
(–2, 4) → (–4, 12) (2, 4) → (4, 12)

$T \circ S$(Parent)
(–2, 3) → (–1, 1) (2, 3) → (3, 1)
(–4, 12) → (–3, 10) (4, 12) → (5, 10)

By the Graph Scale-Change Theorem, the image of the graph of $y = x^2$ under the scale change S: $(x, y) \rightarrow (2x, 3y)$ has equation

$$\frac{y}{3} = \left(\frac{x}{2}\right)^2.$$

By the Graph-Translation Theorem, the image of the graph of this new equation under T: $(x, y) \rightarrow (x + 1, y - 2)$ is

$$\frac{y+2}{3} = \left(\frac{x-1}{2}\right)^2.$$

This is an equation for the image of the graph of $y = x^2$ under the composite $T \circ S$. For instance, the point $D'' = (-3, 10)$ is the image of $D = (-2, 4)$ under $T \circ S$. When $(-3, 10)$ is substituted into the equation $\frac{y+2}{3} = \left(\frac{x-1}{2}\right)^2$, we get $\frac{10+2}{3} = \left(\frac{-3-1}{2}\right)^2$. It checks.

STOP QY

QY

Find the coordinates of the vertex of each parabola on the previous page.

The General Idea

Suppose S is a scale change and T a translation with

$$S(x, y) = (ax, by), \text{ where } a \neq 0 \text{ and } b \neq 0,$$
$$\text{and } T(x, y) = (x + h, y + k).$$

Then the translation image of the scale-change image of (x, y) is

$$\begin{aligned} T \circ S(x, y) &= T(S(x, y)) \\ &= T(ax, by) \\ &= (ax + h, by + k). \end{aligned}$$

Thus, $T \circ S$ maps (x, y) to $(ax + h, by + k)$. That is, if (x', y') is the image of (x, y) under $T \circ S$, then

$$T \circ S(x, y) = (x', y') = (ax + h, by + k).$$

This equation for $T \circ S$ helps to determine how this transformation affects equations. Since

$$x' = ax + h \text{ and } y' = by + k,$$

it follows that $\frac{x' - h}{a} = x$ and $\frac{y' - k}{b} = y$.

This argument proves the following theorem.

Graph-Standardization Theorem

Given a preimage graph described by a sentence in x and y, the following processes yield the same graph:

(1) replacing x by $\frac{x-h}{a}$ and y by $\frac{y-k}{b}$ in the sentence;

(2) applying the scale change $(x, y) \rightarrow (ax, by)$ followed by the translation $(x, y) \rightarrow (x + h, y + k)$ to the preimage graph.

The Graph-Standardization Theorem applies to all relations graphed on a rectangular coordinate grid.

The Graph-Standardization Theorem and Trigonometric Functions

The scale change $(x, y) \rightarrow (ax, by)$ multiplies the period of a sine wave by $|a|$ and its amplitude by $|b|$. The translation $(x, y) \rightarrow (x + h, y + k)$ shifts the image h units horizontally and k units vertically. Combining these two transformations produces the following theorem.

Theorem (Characteristics of a Sine Wave)

The graphs of the functions with equations $\frac{y-k}{b} = \sin\left(\frac{x-h}{a}\right)$ and $\frac{y-k}{b} = \cos\left(\frac{x-h}{a}\right)$, with $a \neq 0$ and $b \neq 0$, have

amplitude $= |b|$, period $= 2\pi|a|$,

phase shift $= h$, and vertical shift $= k$.

Forms of Equations

We call $\frac{y-k}{b} = \sin\left(\frac{x-h}{a}\right)$ or $\frac{y-k}{b} = \cos\left(\frac{x-h}{a}\right)$ the *graph-standardized form* of the equation for a sine or cosine function. When an equation is in this form, you can use the above theorem to determine characteristics of the sine wave.

Example 1

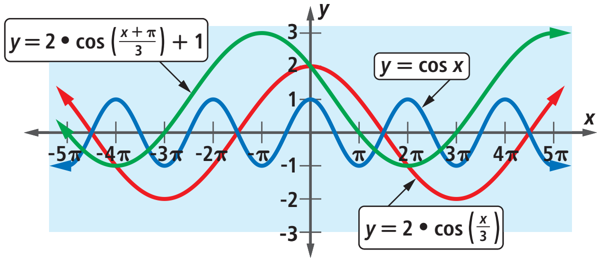
a. Explain how the graph of $\frac{y-1}{2} = \cos\left(\frac{x+\pi}{3}\right)$ is related to the graph of $y = \cos x$.

b. Identify the amplitude, period, vertical shift, and phase shift of this function.

Solution

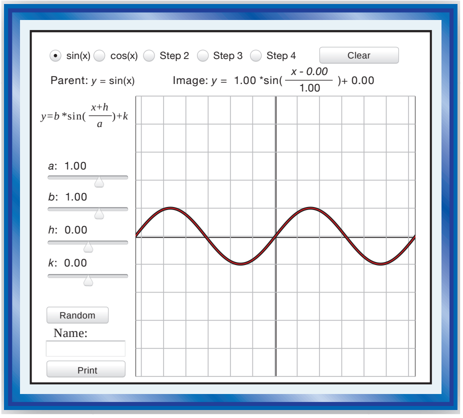
a. The given equation results from the equation $y = \cos x$ by replacing x by $\frac{x-(-\pi)}{3}$ and y by $\frac{y-1}{2}$. Thus the graph of $\frac{y-1}{2} = \cos\left(\frac{x+\pi}{3}\right)$ is the image of the graph of $y = \cos x$ under the scale change $(x, y) \rightarrow (3x, 2y)$ followed by the translation $(x, y) \rightarrow (x - \pi, y + 1)$.

b. From the Characteristics Theorem above, you can determine that the amplitude of $\frac{y-1}{2} = \cos\left(\frac{x+\pi}{3}\right)$ is 2; its period is 6π, the vertical shift is up 1 and the phase shift is $-\pi$.



Check Graph $y = \cos x$, as shown in blue. Next, graph its image under the scale change as shown by the curve drawn in red. Then graph the translation image of the red curve to get the green curve.

Activity



MATERIALS sinusoidal graph application provided by your teacher

Step 1 Open the sinusoidal graph application. The x-axis of the graph is measured in radians. The display shows the parent graph of $y = \sin x$ in black along with another sinusoidal image in red. Move the sliders one at a time to see how each variable affects the image. Change the parent function to the cosine function by clicking the cos(x) button at the top of the screen. Experiment with the sliders again. Record what property of the parent function is affected by each of a, b, h, and k.

Step 2 Click the Step 2 check box button at the top of the screen to create a new sinusoidal image in blue. Note that the parent function is the sine function. Move one slider to scale the red graph to have the same amplitude as the blue graph. Then move another slider to have the red graph coincide with the blue graph. Write the equation of the red graph in $\frac{y-k}{b} = \sin\left(\frac{x-h}{a}\right)$ form, and then simplify $\frac{y-k}{b}$ and $\frac{x-h}{a}$ if possible.

Step 3 Click the Step 3 check box. Note that the parent function has changed to the cosine function. Move a slider to make the red graph have the same period as the blue graph. Then move a slider to translate the red graph to match the blue graph. Write the equation of the red graph in $\frac{y-k}{b} = \cos\left(\frac{x-h}{a}\right)$ form, and then simplify.

Step 4 Click the Step 4 check box. What is the parent function? Move the sliders to make the red graph completely match the blue graph. Write an equation for the red graph in graph-standardized form.

Step 5 Choose a parent function by clicking the sin(x) or cos(x) button at the top of the screen. Press the Random button to generate the graph of a random sinusoidal function. Adjust the sliders for a and b to transform the red graph to have the same shape as the generated graph, then adjust the sliders for k and h to translate your graph to coincide with the generated graph. Record your values of a, b, h, and k, and write the equation in graph-standardized form.

The form $y = A\sin(Bx + C) + D$ or $y = A\cos(Bx + C) + D$ is used in many applications. To convert $\left(\frac{y-1}{2}\right) = \cos\left(\frac{x-\pi}{3}\right)$ to this form, multiply the equation by 2 and add 1 to both sides to get

$$y = 2\cos\left(\frac{x-\pi}{3}\right) + 1.$$

Then, note that this can be rewritten as

$$y = 2\cos\left(\frac{x}{3} - \frac{\pi}{3}\right) + 1,$$

or equivalently,

$$y = 2\cos\left(\frac{1}{3}x - \frac{\pi}{3}\right) + 1.$$

In this form, $A = 2$, $B = \frac{1}{3}$, $C = -\frac{\pi}{3}$, and $D = 1$.

It is useful to be able to convert between forms. To find the characteristics of the sine wave, it helps to rewrite an equation in graph-standardized form.

Example 2

Consider the graph of $y = 2\sin(3x + \pi)$.

a. Describe this graph as the image of the graph of $y = \sin x$ under a composite of transformations.

b. Without graphing, determine the amplitude, period, vertical shift, and phase shift of the sine wave.

Solution

a. Convert the equation into graph-standardized form.

$$\frac{y}{2} = \sin(3x + \pi) = \sin\left(3\left(x + \frac{\pi}{3}\right)\right) = \sin\left(\frac{x - \left(\frac{-\pi}{3}\right)}{\frac{1}{3}}\right)$$

Thus the graph of $y = 2\sin(3x + \pi)$ is the image of the graph of $y = \sin x$ under the scale change $(x, y) \to \left(\frac{1}{3}x, 2y\right)$ followed by the translation $(x, y) \to \left(x - \frac{\pi}{3}, y\right)$.

b. The amplitude of the sine wave is $|2| = 2$. The period is $2\pi\left(\frac{1}{3}\right) = \frac{2\pi}{3}$. There is no vertical shift. The phase shift is $\frac{-\pi}{3}$.

Circular Motion

Trigonometric functions of the forms studied in this lesson arise naturally from situations of circular motion.

Example 3

The first Ferris wheel was built in 1893 for the Columbian Exposition. The radius of the wheel was about 131 feet and its center was about 140 feet above the ground. Find an equation that models the height above ground level of a car at time *x* (in minutes) assuming the wheel is continually rotating at 9 minutes per revolution, and that the car is initially at the maximum height.

Solution Let y be the height of the car that starts at the top. Since the car is initially at maximum height, there is no horizontal translation (phase shift) if the cosine function is used as the parent. So model this situation with the equation $\frac{y - k}{b} = \cos\left(\frac{x - h}{a}\right)$.

Because there is no translation, $h = 0$.

Reinventing the wheel

The original Ferris wheel had 36 cars that could hold 60 passengers each.

(*continued on next page*)

The amplitude is the radius of the wheel, so the amplitude b is 131. Since the center was 140 feet above ground, there is a vertical translation of $k = 140$. The period (one complete revolution or 2π) takes 9 minutes, so if we want time in minutes, we need to shrink the period from 2π to 9. A horizontal shrink with magnitude $\frac{9}{2\pi}$ will do this. Thus $a = \frac{9}{2\pi}$.

Substituting $a = \frac{9}{2\pi}$, $b = 131$, $h = 0$, and $k = 140$ into the equation, we obtain $\frac{y - 140}{131} = \cos\left(\frac{x - 0}{\frac{9}{2\pi}}\right)$, or $y = 131\cos\left(\frac{2\pi}{9}x\right) + 140$.

Check After $2\frac{1}{4}$ minutes, the car should be at the level of the center of the wheel. Is this predicted by the equation? Substitute $2\frac{1}{4}$ for x.

$y = 131 \cdot \cos\left(\frac{2\pi}{9} \cdot 2\frac{1}{4}\right) + 140 = 140$. It checks.

Questions

COVERING THE IDEAS

1. On page 269, the graph of $\frac{y+2}{3} = \left(\frac{x-1}{2}\right)^2$ is shown to be the image of the graph of $y = x^2$ under the composite of a translation and a scale change. Find the image of each point under that composite.
   a. (0, 0) b. (10, 100) c. (–10, 100)
2. What transformations and in what order can be applied to the graph of $y = x^2$ to obtain the graph of $\frac{y-7}{4} = \left(\frac{x+8}{3}\right)^2$?
3. Consider the graph of the function with equation $y = \sin\left(\frac{x-\pi}{3}\right)$.
   a. Give the amplitude, period, and phase shift of the graph.
   b. The graph is the image of the graph of $y = \sin x$ under the composite of what two transformations?
4. Consider the graph of $y = \frac{1}{5}\cos\left(\frac{x+\pi}{6}\right)$.
   a. Give the amplitude, period, and phase shift of the graph.
   b. The graph is the image of the graph of $y = \cos x$ under the composite of what two transformations?

In 5 and 6, an equation of a sine wave is given.
a. Write the equation in graph-standardized form.
b. Find its amplitude, period, phase shift, and vertical shift from the graph of the parent function.

5. $y = 7\sin(\pi x) - 5$
6. $s = 6 + 2\cos(3t + 4)$

In 7–9, refer to Example 3.

7. What are the minimum and maximum heights of the cars?
8. Check that after $4\frac{1}{2}$ minutes you are at the minimum height.
9. Model the height y of the wheel at time x by an equation in the form $\frac{y-k}{b} = \sin\left(\frac{x-h}{a}\right)$, using the sine function as the parent.

APPLYING THE MATHEMATICS

10. Suppose $f(x) = \tan\left(\frac{x+\pi}{4}\right)$.

a. Describe a scale change and translation whose composite maps the graph of $y = \tan x$ onto the graph of $y = f(x)$.

b. Draw a graph of $y = f(x)$ and state its period and phase shift.

11. Create a version of the Graph-Standardization Theorem for the tangent function.

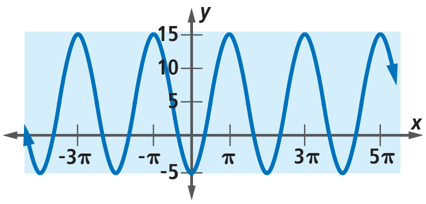
In 12–14, write an equation for the function whose graph has the given characteristics.

12. Parent $y = \cos x$, phase shift $-\frac{\pi}{3}$, period $\frac{\pi}{2}$, and amplitude 9.

13. Parent $y = \sin x$, amplitude 5, period 6π, phase shift $\frac{\pi}{4}$, and vertical shift -1.

14. Parent $y = x^3$, scaled by $S: (x, y) \rightarrow (2x, 3y)$, then translated by $T: (x, y) \rightarrow (x - 1, y + 5)$.

15. The function graphed at the right has maximum value 15, minimum value -5, and period 2π. Write an equation for it.



16. For the sine wave modeled by $y = A \sin(Bx + C) + D$, give formulas in terms of A, B, C, and D for the

a. amplitude. **b.** period. **c.** phase shift.

REVIEW

17. Consider the translation $T: (x, y) \rightarrow \left(x + \frac{\pi}{3}, y + 2\right)$. Find an equation for the image of the graph of $y = |x|$ under T. **(Lesson 3-2)**

18. Identify the following characteristics for the function g defined by $g(x) = A \sin(Bx)$. **(Lesson 4-7)**

a. period **b.** amplitude **c.** domain **d.** range

19. Solve $\sin c = 0$ for $0 \le c \le 6\pi$. **(Lesson 4-5)**

20. A teacher made two forms of a test. The test scores are below.

Form A	93	62	89	77	68	94	73	82	85	76	83	79
Form B	65	87	71	76	67	87	76	81	77	82	62	78

Compared to the other students who took the same test, who performed better, the student with an 85 on Form A or the student with an 82 on Form B? **(Lesson 3-9)**

21. Simplify. **(Previous Course)**

a. $\frac{10^8}{10^3}$ **b.** $\frac{6 \cdot 10^5}{2 \cdot 10^{-3}}$ **c.** $\frac{2.8 \cdot 10^{-2}}{1.4 \cdot 10^3}$ **d.** $\frac{a \cdot 10^m}{b \cdot 10^n}$

EXPLORATION

22. Does it make a difference whether you translate a graph first and then scale the image, or scale the graph first and then translate the image? Defend your answer with at least two examples.

QY ANSWER

(0, 0), (0, 0), (1, –2)